

Thin shell model revisited

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Abstract

We reconsider some fundamental problems of the thin shell model. First, we point out that the “cut and paste” construction does not guarantee a well-defined manifold because there is no overlap of coordinates across the shell. When one requires that the spacetime metric across the thin shell is continuous, it also provides a way to specify the tangent space and the manifold. Other authors have shown that this specification leads to the conservation laws when shells collide. On the other hand, the well-known areal radius r seems to be a perfect coordinate covering all regions of a spherically symmetric spacetime. However, we show by simple but rigorous arguments that r fails to be a coordinate covering a neighborhood of the thin shell if the metric across the shell is continuous. When two spherical shells collide and merge into one, we show that it is possible that r remains to be a good coordinate and the conservation laws hold. To make this happen, different spacetime regions divided by the shells must be glued in a specific way such that some constraints are satisfied. We compare our new construction with the old one by solving constraints numerically.

1 Introduction

Since the pioneering work by Israel [1], the thin shell model has been extensively studied and has a wide application in gravitational collapse, cosmology, wormhole theory, etc. Such a model is important because it is an idealization of the real matter distribution and has given many interesting solutions in general relativity and alternative gravity theories.

Despite great successes of this model, some fundamental problems still remain to be answered. It is well known that a thin shell is a three dimensional hypersurface (in four dimension spacetimes) which can be constructed by the “cut and paste” approach [2]. Let M_1 and M_2 be two distinct spacetimes with

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coordinates $\{x_1^\mu\}$ and $\{x_2^\mu\}$. We can assign a metric $g_{ab}(x_i^\mu)$ to M_i , $i = 1, 2$. Suppose that each manifold is bounded by a hypersurface Σ_i . If we wish to unify the two spacetimes, it is natural to glue them by identifying their boundaries, i.e., the new spacetime $M = M_1 \cup M_2$ connects the two distinct spacetimes at the hypersurface $\Sigma \equiv \Sigma_1 = \Sigma_2$ (see Fig. 1). There is always a discontinuity of the extrinsic curvature across Σ which is related to the surface matter distribution on Σ . So the first derivative of the metric is discontinuous at Σ , which leads to the famous junction conditions [1]. It is also a convention to require that the induced metric h_{ab} is continuous. However, there is no general requirement on the continuity of g_{ab} across Σ .

Before we go further, we should notice that any metric is defined on a manifold. An overlooked question is: have we had a well-defined manifold by the above construction? If we go through the general properties of manifold, we will find immediately that the answer is “not yet”. A manifold allows to be covered by more than one coordinate system. But neither x_1^μ nor x_2^μ covers a neighborhood of a point on Σ . Although both coordinate systems give coordinates on Σ , their overlap is only a three dimensional region, not an open set of the manifold [3]. An obvious consequence is that the tangent space of each point on Σ is not uniquely defined by the construction so far. Before we glue M_1 and M_2 together, we only have two “half tangent spaces” at their boundaries. Identifying the boundaries does not give a unified tangent space for each point at the boundary. There is an ambiguity for each tangent vector $u_1^a \in V_1$ at the boundary to find its “other half” $u_2^a \in V_2$ satisfying $u_1^a + u_2^a = 0$, where V_1 and V_2 are the tangent spaces of M_1 and M_2 , respectively. There are two equivalent ways to fix the ambiguity. First, we can extend $\{x_1^\mu\}$ to M_2 such that there is a four dimensional overlap of $\{x_1^\mu\}$ and $\{x_2^\mu\}$. By this way, M is a well-defined manifold. Second, Note that the three dimensional tangent space of Σ has no ambiguity by construction. Thus, we only need to assign one transversal vector u_1^a in V_1 with a negative transversal vector u_2^a in V_2 such that $u_1^a + u_2^a = 0$ (see Fig. 1). Then any other vector is uniquely assigned a negative vector by the addition rule.

To be definitive, we assume that Σ is timelike and the induced metric is given by

$$h_{ab} = g_{ab}^i + n_a^i n_b^i, \quad i = 1, 2, \quad (1)$$

where n_a^i is the spacelike normal of Σ . Since h_{ab} is the same from both sides, we see immediately that the spacetime metric g_{ab} is continuous if and only if

$$n_a^1 = n_a^2 \quad (2)$$

By our argument above, Eq. (2) also uniquely fixes the tangent space of any point on Σ . In spherically symmetric spacetimes, if a few shells collide, it has been shown by Langlois, Maeda and Wands (LMW) [4] that this identification leads to the conservation of energy and momentum at the collision point. The LMW method has been further applied to bubble and brane collisions [6]-[8].

In a spherically symmetric spacetime, the radial coordinate r is the areal radius of the sphere formed by the $SO(3)$ isometry [3]. So r is a well-defined

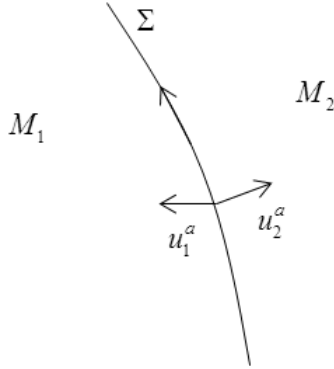


Figure 1: M_1 and M_2 are connected at Σ . For each u_1^a , one needs to specify a u_2^a such that $u_1^1 + u_2^a = 0$.

function and seems to be the only natural coordinate covering different regions divided by the shells. However, we show in section 2 that if the metric is continuous across the shell, r is no longer a good coordinate for points on the shell. Therefore, if r is a good coordinate, we must choose other identifications which break down the continuity of metrics and generally violate the conservation laws as well. In section 3, we consider the simplest collision: two shells merge into one after they collide. we derive some constraint equations such that r remains to be a good coordinate and the conservation laws hold. By imposing appropriate initial conditions, we find that these equations are solvable at least numerically.

2 One spherical thin shell and the r coordinate

We consider a spherical shell Σ moving in a spherical spacetime. The coordinates on the two sides of the shell are labeled by (t_1, r_1) and (t_2, r_2) , where we have dropped the (θ, ϕ) coordinates for simplicity (see Fig. 2).

The metrics on both sides are in the form

$$ds_i^2 = -f_i(r)dt^2 + f_i^{-1}(r)dr^2 + r^2d\Omega^2 \quad (3)$$

where $i = 1, 2$ and in the Schwarzschild case

$$f_i(r) = 1 - \frac{2M_i}{r} \quad (4)$$

Note that $r_1 = r_2$ by continuity, but t is discontinuous across the shell. We may write the four-velocity of the shell as

$$u^a = \dot{t}_i \left(\frac{\partial}{\partial t_i} \right)^a + \dot{r} \left(\frac{\partial}{\partial r_i} \right)^a \quad (5)$$

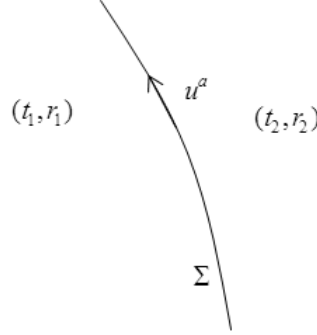


Figure 2: A spherical shell Σ moving with four-velocity u^a .

Note that we have used \dot{r} instead of \dot{r}_i because $r_1 = r_2$. The normalization condition $g_{ab}u^a u^b = -1$ yields

$$\dot{t}_i = \pm \sqrt{\frac{f_i + \dot{r}^2}{f_i^2}} \quad (6)$$

The normal vector of Σ is of the form

$$n_i^a = \frac{\dot{r}}{f_i(r)} \left(\frac{\partial}{\partial t_i} \right)^a + \sqrt{\dot{r}^2 + f_i} \left(\frac{\partial}{\partial r_i} \right)^a \quad (7)$$

Now we have three orthogonal and normal tetrads related by the following Lorentz transformation [4]

$$\begin{pmatrix} u^a \\ n^a \end{pmatrix} = \Lambda(\alpha_i) \begin{pmatrix} \sqrt{\frac{1}{f_i}} \left(\frac{\partial}{\partial t_i} \right)^a \\ \sqrt{f_i} \left(\frac{\partial}{\partial r_i} \right)^a \end{pmatrix} \quad (8)$$

where

$$\Lambda(\alpha) = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \quad (9)$$

and

$$\alpha_i = \sinh^{-1} \frac{\dot{r}}{\sqrt{f_i}} \quad (10)$$

Therefore,

$$\begin{pmatrix} \sqrt{\frac{1}{f_2}} \left(\frac{\partial}{\partial t_2} \right)^a \\ \sqrt{f_2} \left(\frac{\partial}{\partial r_2} \right)^a \end{pmatrix} = \Lambda(\alpha_2 - \alpha_1) \begin{pmatrix} \sqrt{\frac{1}{f_1}} \left(\frac{\partial}{\partial t_1} \right)^a \\ \sqrt{f_1} \left(\frac{\partial}{\partial r_1} \right)^a \end{pmatrix} \quad (11)$$

We should emphasize that the continuity of metric is crucial to derive this formula. However, we show now that this treatment is inconsistent with the assumption that r is a good coordinate.

As we have mentioned above, the two sets of coordinates $\{t_1, r_1\}$ and $\{t_2, r_2\}$ do not have a four-dimensional overlap in the neighborhood of Σ . We need first extend the coordinates smoothly such that they have a four-dimensional overlap region O_p where $p \in \Sigma$. If r is a good coordinate everywhere, we should have

$$r_1 = r_2 = r \quad (12)$$

in O_p .

Then we can write down the transformation at p

$$\left(\frac{\partial}{\partial t_2}\right)^a = \frac{\partial t_1}{\partial t_2} \left(\frac{\partial}{\partial t_1}\right)^a + \frac{\partial r_1}{\partial t_2} \left(\frac{\partial}{\partial r_1}\right)^a \quad (13)$$

The second term vanishes due to $r_1 = r_2$. So

$$\left(\frac{\partial}{\partial t_2}\right)^a = \frac{\partial t_1}{\partial t_2} \left(\frac{\partial}{\partial t_1}\right)^a \quad (14)$$

Similarly,

$$\begin{aligned} \left(\frac{\partial}{\partial r_2}\right)^a &= \frac{\partial t_1}{\partial r_2} \left(\frac{\partial}{\partial t_1}\right)^a + \frac{\partial r_1}{\partial r_2} \left(\frac{\partial}{\partial r_1}\right)^a \\ &= \frac{\partial t_1}{\partial r_2} \left(\frac{\partial}{\partial t_1}\right)^a + \left(\frac{\partial}{\partial r_1}\right)^a \end{aligned} \quad (15)$$

Note that $\alpha_1 \neq \alpha_2$ due to $f_1 \neq f_2$. Therefore, Eq. (11) indicates that $\left(\frac{\partial}{\partial t_1}\right)^a$ is not parallel to $\left(\frac{\partial}{\partial t_2}\right)^a$. This contradicts Eq. (14).

Another quick way to see the breakdown of r is to notice that the $\theta\theta$ component of the extrinsic curvature of Σ is given by [5]

$$K_{\theta\theta} = rn^r = r\sqrt{f + \dot{r}^2} \quad (16)$$

which is obviously discontinuous across the shell. This discontinuity leads to the junction condition. Note that

$$n^r = n^a(dr)_a \quad (17)$$

So if $n_1^a = n_2^a$, the discontinuity of n^r indicates that r is not a qualified function in any neighborhood of $p \in \Sigma$.

3 Collision of shells and the conservation laws

In this section, we shall match different sides of shells in a way such that r is a good coordinate across all shells. Then we shall discuss the conservation laws when shells collide. For simplicity, we consider the collision of two shells. After the collision, they merge as one shell.

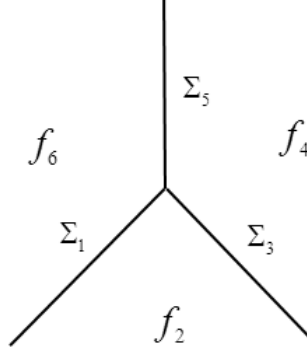


Figure 3: Two shells collide and stick together.

3.1 Matching conditions

As shown in Fig. 3, Σ_1 and Σ_3 represent two shells before the collision and Σ_5 represents the shell after the collision. The spacetime is then divided into three parts covered by coordinates $\{t_i, r_i\}$, $i = 2, 4, 6$. Applying Eqs. (14) and (15) to Σ_1 , we have

$$\left(\frac{\partial}{\partial t_2}\right)^a = T_1 \left(\frac{\partial}{\partial t_6}\right)^a \quad (18)$$

$$\left(\frac{\partial}{\partial r_2}\right)^a = R_1 \left(\frac{\partial}{\partial t_6}\right)^a + \left(\frac{\partial}{\partial r_6}\right)^a \quad (19)$$

where $T_1 = \frac{\partial t_6}{\partial t_2}$ and $R_1 = \frac{\partial t_6}{\partial r_2}$.

Similarly, on Σ_3 and Σ_5 we have

$$\left(\frac{\partial}{\partial t_4}\right)^a = T_3 \left(\frac{\partial}{\partial t_2}\right)^a \quad (20)$$

$$\left(\frac{\partial}{\partial r_4}\right)^a = R_3 \left(\frac{\partial}{\partial t_2}\right)^a + \left(\frac{\partial}{\partial r_2}\right)^a \quad (21)$$

$$\left(\frac{\partial}{\partial t_6}\right)^a = T_5 \left(\frac{\partial}{\partial t_4}\right)^a \quad (22)$$

$$\left(\frac{\partial}{\partial r_6}\right)^a = R_5 \left(\frac{\partial}{\partial t_4}\right)^a + \left(\frac{\partial}{\partial r_4}\right)^a \quad (23)$$

Combining Eqs. (22), (20) and (18), we have

$$\left(\frac{\partial}{\partial t_6}\right)^a = T_5 T_3 T_1 \left(\frac{\partial}{\partial t_6}\right)^a \quad (24)$$

while Eqs. (23), (21) and (19) give

$$\left(\frac{\partial}{\partial r_6}\right)^a = R_5 T_3 T_1 \left(\frac{\partial}{\partial t_6}\right)^a + R_3 T_1 \left(\frac{\partial}{\partial t_6}\right)^a + R_1 \left(\frac{\partial}{\partial t_6}\right)^a + \left(\frac{\partial}{\partial r_6}\right)^a \quad (25)$$

Therefore,

$$T_5 T_3 T_1 = 1 \quad (26)$$

$$R_5 T_3 T_1 + R_3 T_1 + R_1 = 0 \quad (27)$$

On the other hand, the four-velocity of Σ_1 reads

$$u_1^a = \dot{t}_{16} \left(\frac{\partial}{\partial t_6}\right)^a + \dot{r}_1 \left(\frac{\partial}{\partial r_6}\right)^a = \dot{t}_{12} \left(\frac{\partial}{\partial t_2}\right)^a + \dot{r}_1 \left(\frac{\partial}{\partial r_2}\right)^a \quad (28)$$

where

$$\dot{t}_{16} = \sqrt{\frac{f_6 + \dot{r}_1^2}{f_6^2}}, \quad \dot{t}_{12} = \sqrt{\frac{f_2 + \dot{r}_1^2}{f_2^2}} \quad (29)$$

Then junction condition reads

$$m_1 = r_1 \left(\sqrt{f_6 + \dot{r}_1^2} - \sqrt{f_2 + \dot{r}_1^2} \right) \quad (30)$$

Substituting Eqs. (18) and (19) into Eq. (28) yields

$$\dot{t}_{16} \left(\frac{\partial}{\partial t_6}\right)^a + \dot{r}_1 \left(\frac{\partial}{\partial r_6}\right)^a = \dot{t}_{12} T_1 \left(\frac{\partial}{\partial t_6}\right)^a + \dot{r}_1 R_1 \left(\frac{\partial}{\partial t_6}\right)^a + \dot{r}_1 \left(\frac{\partial}{\partial r_6}\right)^a \quad (31)$$

which gives

$$\dot{t}_{16} = \dot{t}_{12} T_1 + \dot{r}_1 R_1 \quad (32)$$

Similarly, On Σ_3 and Σ_5 , we have

$$\dot{t}_{32} = \dot{t}_{34} T_3 + \dot{r}_3 R_3 \quad (33)$$

$$\dot{t}_{54} = \dot{t}_{56} T_5 + \dot{r}_5 R_5 \quad (34)$$

3.2 Conservation of energy and momentum

The conservation law is

$$m_1 u_1^a + m_3 u_3^a - m_5 u_5^a = 0 \quad (35)$$

Note that

$$u_1^a = \dot{t}_{12} \left(\frac{\partial}{\partial t_2}\right)^a + \dot{r}_1 \left(\frac{\partial}{\partial r_2}\right)^a \quad (36)$$

$$u_3^a = \dot{t}_{32} \left(\frac{\partial}{\partial t_2} \right)^a + \dot{r}_3 \left(\frac{\partial}{\partial r_2} \right)^a \quad (37)$$

$$\begin{aligned} u_5^a &= \dot{t}_{54} \left(\frac{\partial}{\partial t_4} \right)^a + \dot{r}_5 \left(\frac{\partial}{\partial r_4} \right)^a \\ &= \dot{t}_{54} T_3 \left(\frac{\partial}{\partial t_2} \right)^a + \dot{r}_5 \left[R_3 \left(\frac{\partial}{\partial t_2} \right)^a + \left(\frac{\partial}{\partial r_2} \right)^a \right] \\ &= (\dot{t}_{54} T_3 + \dot{r}_5 R_3) \left(\frac{\partial}{\partial t_2} \right)^a + \dot{r}_5 \left(\frac{\partial}{\partial r_2} \right)^a \end{aligned} \quad (38)$$

where Eqs. (20) and (21) have been used.

So the conservation law yields

$$m_1 \dot{r}_1 + m_3 \dot{r}_3 - m_5 \dot{r}_5 = 0 \quad (39)$$

$$m_1 \dot{t}_{12} + m_3 \dot{t}_{32} - m_5 (\dot{t}_{54} T_3 + \dot{r}_5 R_3) = m_1 \dot{t}_{12} + m_3 \dot{t}_{32} - m_5 \dot{t}_{52} = 0 \quad (40)$$

where we have used

$$\dot{t}_{52} \equiv \frac{\partial t_2}{\partial \tau} \Big|_5 = \frac{\partial t_2}{\partial t_4} \frac{\partial t_4}{\partial \tau} + \frac{\partial t_2}{\partial r_4} \dot{r}_5 = T_3 \dot{t}_{54} + R_3 \dot{r}_5 \quad (41)$$

in the last step.

Eq. (39) is the r component of Eq. (35), which is particularly simple. The corresponding equations in the LMW method are not equivalent to our Eqs. (39) and (40), although the vector form (35) of the conservation law is the same in both methods.

3.3 Solving equations

Now we have 15 independent variables:

$$\begin{aligned} &m_1, \dot{r}_1, m_3, \dot{r}_3, m_5, \dot{r}_5 \\ &f_2, f_4, f_6 \\ &T_1, T_3, T_5, R_1, R_3, R_5 \end{aligned}$$

while there are 10 equations: 3 junction conditions, Eqs. (26) and (27), Eqs. (32),(33),(34), plus two equations of conservation of energy and momentum (Eq. (39) and (40)). We may set initial data: f_2, f_4, f_6, m_1, m_3 , then the rest variables can be solved.

3.4 Numerical results

We take

$$f_6 = 0.7, f_2 = 0.6, f_4 = 0.5, \dot{r}_1 = 1, \dot{r}_3 = 2, r = 10. \quad (42)$$

Then the ten equations mentioned above give rise to the following numerical solutions:

$$m_1 = 0.3893, m_3 = 0.2344, m_5 = 0.663494, \dot{r}_5 = 1.2933 \quad (43)$$

$$T_1 = 0.9194, T_3 = 0.9208, T_5 = 1.1811$$

$$R_1 = -0.0758, R_3 = -0.1661, R_5 = 0.2699 \quad (44)$$

Note that the solutions of T_i and R_i tell us how the manifold is constructed. As we have discussed, this match of manifold differs from the LMW treatment. However, the initial conditions of Eq. (42) are exactly needed for the LMW method (see Appendix A). It is not surprising that the two methods give rise to different solutions for m_5 and \dot{r}_5 (see Eq. (43) and Eq. (66)) because the matching conditions are different.

4 Conclusions

In this paper, some fundamental problems of the thin shell model have been reconsidered and clarified. To make the thin shell spacetime a well-defined manifold, some extra conditions need to be imposed. Some authors have proven that the continuity of the metric across the shell leads to the conservation of energy and momentum in spherically symmetric spacetimes. However, we show that in this treatment, the areal radius r is no longer a coordinate covering a neighborhood of the shell. We have then proposed a new matching technique such that the conservation law and the coordinate r are both preserved. In the case that two shells collide and merge into one shell, we have shown that the initial conditions that needed to solve all the equations are exactly the same as in the LMW method. Our work suggests that spacetimes containing thin shells can be matched in different ways and the conservation laws can still be preserved.

Acknowledgements

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A Review of LMW mechanism and numerical calculation

In this appendix, we review LMW's treatment and apply it to the case in section 3. With the same initial conditions, the numerical computation shows that the two methods gives different results.

A.1 One shell

The junction condition of one shell is

$$\sqrt{f_1 + \dot{r}^2} - \sqrt{f_2 + \dot{r}^2} = \rho r \quad (45)$$

where r increases from region 1 to region 2 and ρ is the surface density of the shell. Let

$$\sinh \alpha_i = \frac{\dot{r}}{\sqrt{f_i}} \quad (46)$$

we have

$$\sqrt{f_i + \dot{r}^2} = \sqrt{f_i} \sqrt{1 + \sinh^2 \alpha_i} = \sqrt{f_i} \cosh \alpha_i \quad (47)$$

Let

$$\tilde{\rho} = \rho r \quad (48)$$

Then

$$\sqrt{f_1} \cosh \alpha_1 - \sqrt{f_2} \cosh \alpha_2 = \tilde{\rho} \quad (49)$$

i.e.,

$$\frac{1}{2} \sqrt{f_1} e^{\alpha_1} + \frac{1}{2} \sqrt{f_1} e^{-\alpha_1} - \frac{1}{2} \sqrt{f_2} e^{\alpha_2} - \frac{1}{2} \sqrt{f_2} e^{-\alpha_2} = \tilde{\rho} \quad (50)$$

Eq. (46) leads to

$$\frac{e^{\alpha_1} - e^{-\alpha_1}}{e^{\alpha_2} - e^{-\alpha_2}} = \frac{\sqrt{f_2}}{\sqrt{f_1}} \quad (51)$$

So Eq. (50) may be written as

$$\sqrt{f_1} e^{\alpha_1} - \sqrt{f_2} e^{\alpha_2} = \tilde{\rho} \quad (52)$$

$$\sqrt{f_1} e^{-\alpha_1} - \sqrt{f_2} e^{-\alpha_2} = \tilde{\rho} \quad (53)$$

A.2 Three shells

We still consider the collision of two shells as shown in Fig. 3. Applying Eq. (46) to each shell, we have

$$\sinh \alpha_{16} = \frac{\dot{r}_1}{\sqrt{f_6}}, \quad \sinh \alpha_{12} = \frac{\dot{r}_1}{\sqrt{f_2}} \quad (54)$$

$$\sinh \alpha_{32} = \frac{\dot{r}_3}{\sqrt{f_2}}, \quad \sinh \alpha_{34} = \frac{\dot{r}_3}{\sqrt{f_4}} \quad (55)$$

$$\sinh \alpha_{54} = \frac{\dot{r}_5}{\sqrt{f_4}}, \quad \sinh \alpha_{56} = \frac{\dot{r}_5}{\sqrt{f_6}} \quad (56)$$

Applying Eq. (52) to each shell yields

$$\tilde{\rho}_1 = \sqrt{f_6}e^{\alpha_{16}} - \sqrt{f_2}e^{\alpha_{12}} \quad (57)$$

$$\tilde{\rho}_3 = \sqrt{f_2}e^{\alpha_{32}} - \sqrt{f_4}e^{\alpha_{34}} \quad (58)$$

$$\tilde{\rho}_5 = \sqrt{f_4}e^{\alpha_{54}} - \sqrt{f_6}e^{\alpha_{56}} \quad (59)$$

Note that $\tilde{\rho}_5$ given above has a sign difference from what we gave previously because it corresponds to the shell after the collision.

The consistency condition is given by [4]

$$\alpha_{16} - \alpha_{12} + \alpha_{32} - \alpha_{34} + \alpha_{54} - \alpha_{56} = 0 \quad (60)$$

If we define

$$\alpha_{ij} = -\alpha_{ji} \quad (61)$$

The condition becomes

$$\alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{45} + \alpha_{56} + \alpha_{61} = 0 \quad (62)$$

For f_2 ,

$$\begin{aligned} & \tilde{\rho}_1 e^{\alpha_{23}} e^{\alpha_{34}} e^{\alpha_{45}} e^{\alpha_{56}} e^{\alpha_{61}} + \tilde{\rho}_3 e^{\alpha_{23}} + \tilde{\rho}_5 e^{\alpha_{23}} e^{\alpha_{34}} e^{\alpha_{45}} \\ &= \sqrt{f_6} e^{\alpha_{16}} e^{\alpha_{23}} e^{\alpha_{34}} e^{\alpha_{45}} e^{\alpha_{56}} e^{\alpha_{61}} - \sqrt{f_6} e^{\alpha_{56}} e^{\alpha_{23}} e^{\alpha_{34}} e^{\alpha_{45}} + \dots = 0 \end{aligned} \quad (63)$$

where we have used Eq. (62). Such relations can be written as

$$\tilde{\rho}_1 e^{\pm\alpha_{21}} + \tilde{\rho}_3 e^{\pm\alpha_{23}} + \tilde{\rho}_5 e^{\pm\alpha_{25}} = 0 \quad (64)$$

where $\alpha_{25} = \alpha_{23} + \alpha_{34} + \alpha_{45}$. The other two relations can be obtained by replacing 2 by 4 and 6. It is also easy to get

$$\tilde{\rho}_1 \gamma_{21} + \tilde{\rho}_3 \gamma_{23} + \tilde{\rho}_5 \gamma_{25} = 0 \quad (65)$$

where $\gamma_{ij} = \cosh \alpha_{ij}$. This is the energy conservation law [4].

A.3 Numerical results

We use the same initial data as given in Eq. (42). Then Eqs. (54) and (55) can be solved directly. Combining Eq. (56) and Eq. (60), one can obtain $\alpha_{54} = 1.3876$ and $\alpha_{56} = 1.2421$. Then we have

$$m_5 = 0.6506, \quad \dot{r}_5 = 1.3278 \quad (66)$$

which is different from the previous results.

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